

# RESTRICTION OF AVERAGING OPERATORS TO ALGEBRAIC VARIETIES OVER FINITE FIELDS

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**ABSTRACT.** We study  $L^p \rightarrow L^r$  estimates for restricted averaging operators related to algebraic varieties  $V$  of  $d$ -dimensional vector spaces over finite fields  $\mathbb{F}_q$  with  $q$  elements. We observe properties of both the Fourier restriction operator and the averaging operator over  $V \subset \mathbb{F}_q^d$ . As a consequence, we obtain optimal results on the restricted averaging problems for spheres and paraboloids in dimensions  $d \geq 2$ , and cones in odd dimensions  $d \geq 3$ . In addition, when the variety  $V$  is a cone lying in an even dimensional vector space over  $\mathbb{F}_q$  and  $-1$  is a square number in  $\mathbb{F}_q$ , we also obtain sharp estimates except for two endpoints.

## 1. INTRODUCTION

Over the past decade there has been a lot of interest in developing harmonic analysis over finite fields. Mockenhaupt and Tao in [16] initially studied the finite field restriction problem. The finite field Kakeya problem was posed by Wolff in [17]. These two problems are considered as central problems in Euclidean harmonic analysis. Amazingly, Dvir in [4] recently found a simple proof of the finite field Kakeya conjecture, wherein he invoked the polynomial method. There are some serious difficulties in adapting the Dvir's proof to the Euclidean Kakeya problem. On the other hand the polynomial method plays an important role in improving some problems in harmonic analysis. For instance, Guth in [7] used it to obtain an improvement on the Euclidean restriction problem (see also [6, 8]). This example demonstrates that finite field analogues can be useful in developing methods for Euclidean analogs.

Finite fields can be an efficient method by which one can introduce a problem to mathematicians in other fields. This is mainly due to finite fields possessing a relatively simple structure. Furthermore, problems in finite fields are closely related to other mathematical subjects such as algebraic geometry, additive number theory, or combinatorics. For these reasons, analysis problems in finite fields have received much attention in the last few decades (see, for example, [2, 5, 9, 14, 15]).

In this paper we study a hybrid of the averaging operators and restriction operators in the finite field setting. Roughly speaking, given an algebraic variety

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2000 *Mathematics Subject Classification.* Primary: 42B05 ; Secondary 11T23 .

*Key words and phrases.* Algebraic curves, finite fields, restricted averaging operators.

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This research was supported by the research grant of Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2015R1A1A1A05001374).

$V \subset \mathbb{F}_q^d$  and an operator  $T$  acting on functions  $f : \mathbb{F}_q^d \rightarrow \mathbb{C}$ , a new operator  $T_V$  can be defined by restricting  $Tf$  to the variety  $V$ . Then a natural question is to determine the boundedness of the restricted operator  $T_V$ . Note that when  $Tf$  is the Fourier transform of  $f$ , this problem becomes the Fourier restriction problem. While it is possible to study the extended Fourier restriction problem for various operators, we shall focus on studying the problem for averaging operators  $T$  over algebraic varieties  $V \subset \mathbb{F}_q^d$ . We call this the restricted averaging problem to  $V$ . It was observed in [12] that optimal results can be obtained if the variety  $V$  is any curve on two dimensions which does not contain a line. In this paper we extend the work to higher dimensions.

**1.1. Discrete Fourier analysis.** We begin by reviewing definitions and notation. Let  $\mathbb{F}_q^d$  be a  $d$ -dimensional vector space over the finite field  $\mathbb{F}_q$  with  $q$  elements. Throughout this paper we always assume that  $q$  is the power of an odd prime. We endow  $\mathbb{F}_q^d$  with the normalized counting measure  $dx$ . We shall use  $(\mathbb{F}_q^d, dx)$  to indicate the  $d$ -dimensional vector space with the normalized counting measure  $dx$ . In order to indicate the dual space of  $(\mathbb{F}_q^d, dx)$ , we shall use  $(\mathbb{F}_q^d, dm)$  which is endowed with the counting measure  $dm$ . Thus,

$$\int_{\mathbb{F}_q^d} f(x) dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(x) \quad \text{for } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C},$$

and

$$\int_{\mathbb{F}_q^d} g(m) dm = \sum_{m \in \mathbb{F}_q^d} g(m) \quad \text{for } g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}.$$

Let  $\chi$  denote a fixed nontrivial additive character of  $\mathbb{F}_q$ . Our results will be independent of our choice of character.

Given a function  $g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}$ , the Fourier transform of  $g$  is given by

$$\widehat{g}(x) = \int_{\mathbb{F}_q^d} \chi(-x \cdot m) g(m) dm := \sum_{m \in \mathbb{F}_q^d} \chi(-x \cdot m) g(m) \quad \text{for } x \in (\mathbb{F}_q^d, dx).$$

On the other hand, if  $f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$ , then the inverse Fourier transform of  $f$  is given by

$$f^\vee(m) = \int_{\mathbb{F}_q^d} \chi(m \cdot x) f(x) dx := \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) f(x) \quad \text{for } m \in (\mathbb{F}_q^d, dm).$$

Recall that by orthogonality we have

$$\sum_{x \in \mathbb{F}_q^d} \chi(m \cdot x) = \begin{cases} 0 & \text{if } m \neq (0, \dots, 0) \\ q^d & \text{if } m = (0, \dots, 0), \end{cases}$$

where  $m \cdot x$  is the usual dot-product. By the above orthogonality relation of  $\chi$ , we obtain Plancherel's theorem which states  $\|f^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = \|f\|_{L^2(\mathbb{F}_q^d, dx)}$ . Namely, Plancherel's theorem yields

$$\sum_{m \in \mathbb{F}_q^d} |f^\vee(m)|^2 = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} |f(x)|^2.$$

Given functions  $f, h : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$ , the convolution  $f * h$  is defined as

$$f * h(y) = \int_{\mathbb{F}_q^d} f(y-x)h(x) dx = \frac{1}{q^d} \sum_{x \in \mathbb{F}_q^d} f(y-x)h(x) \quad \text{for } y \in (\mathbb{F}_q^d, dx).$$

Notice that  $(f * h)^\vee(m) = f^\vee(m)h^\vee(m)$  for  $m \in (\mathbb{F}_q^d, dm)$ .

Given an algebraic variety  $V \subset (\mathbb{F}_q^d, dx)$ , we endow  $V$  with the normalized surface measure  $\sigma$  which is defined by the relation

$$\int_V f(x) d\sigma(x) := \frac{1}{|V|} \sum_{x \in V} f(x),$$

where  $|V|$  denotes the cardinality of the set  $V$  and  $f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}$ . Notice that  $d\sigma(x)$  is replaced by  $(q^d/|V|) V(x) dx$ . Throughout this paper, we shall identify the set  $V \subset \mathbb{F}_q^d$  with the characteristic function  $\chi_V$  on  $V$ , so that  $V(x) = 1$  for  $x \in V$  and  $V(x) = 0$  otherwise.

**1.2. Definition of the restricted averaging operator.** With the above notation the averaging operator  $A$  is defined by

$$Af(y) = f * \sigma(y) = \int_V f(y-x) d\sigma(x) := \frac{1}{|V|} \sum_{x \in V} f(y-x),$$

where  $f$  and  $Af$  are defined on  $(\mathbb{F}_q^d, dx)$ . The averaging operator  $A$  was initially introduced for finite fields by Carbery-Stone-Wright [3]. Sharp  $L^p \rightarrow L^r$  estimates of the averaging operator  $A$  were obtained in the case when  $V$  is the sphere, the paraboloid, or the cone ([11, 13]). Now, we consider a restricted operator  $A_V$  defined by restricting  $Af = f * \sigma$  to the algebraic variety  $V$ . Namely, we have  $A_V f = Af|_V$ . We call the operator  $A_V$  the restricted averaging operator to the algebraic variety  $V \subset (\mathbb{F}_q^d, dx)$ . The main purpose of this paper is to study  $L^p \rightarrow L^r$  estimates for the restricted averaging operator  $A_V$ .

*Problem 1.1.* (Restricted averaging problem) Let  $\sigma$  be the normalized surface measure on the variety  $V \subset (\mathbb{F}_q^d, dx)$ . For  $1 \leq p, r \leq \infty$ , we define  $A_V(p \rightarrow r)$  as the smallest constant  $C > 0$  such that the estimate

$$(1.1) \quad \|f * \sigma\|_{L^r(V, \sigma)} \leq C \|f\|_{L^p(\mathbb{F}_q^d, dx)}$$

holds for all functions  $f : \mathbb{F}_q^d \rightarrow \mathbb{C}$ . The quantity  $A_V(p \rightarrow r)$  may depend on  $q$ , the cardinality of the underlying finite field  $\mathbb{F}_q$ . The restricted averaging problem to  $V$  is to determine all pairs  $(p, r)$  such that  $1 \leq p, r \leq \infty$  and  $A_V(p \rightarrow r)$  is independent of the field size  $q$ .

For positive numbers  $A$  and  $B$ , we use  $A \lesssim B$  if there is a constant  $C > 0$  independent of the field size  $q$  such that  $A \leq CB$ . We also use  $A \sim B$  to indicate that  $A \lesssim B$  and  $B \lesssim A$ . In this setting, the restricted averaging problem is to find all pairs  $(p, r)$  such that  $1 \leq p, r \leq \infty$  and  $A_V(p \rightarrow r) \lesssim 1$ . Here, we again stress that the implicit constant in  $\lesssim$  is allowed to depend on  $d, p, r$  but it must be independent of  $q = |\mathbb{F}_q|$ .

**Remark 1.2.** Since  $\|1\|_{L^s(V, \sigma)} = 1 = \|f\|_{L^s(\mathbb{F}_q^d, dx)}$  for all  $1 \leq s \leq \infty$ , we see from Hölder's inequality that

$$A_V(p_2 \rightarrow r) \leq A_V(p_1 \rightarrow r) \quad \text{for } 1 \leq p_1 \leq p_2 \leq \infty$$

and

$$A_V(p \rightarrow r_1) \leq A_V(p \rightarrow r_2) \quad \text{for } 1 \leq r_1 \leq r_2 \leq \infty$$

which will allow us to reduce the analysis below to certain endpoint estimates.

As usual, we denote by  $A_V^*$  the adjoint operator of the restricted averaging operator to  $V$ . Since

$$\langle A_V f, h \rangle_{L^2(V, \sigma)} = \langle f, A_V^* h \rangle_{L^2(\mathbb{F}_q^d, dx)},$$

the adjoint operator  $A_V^*$  of  $A_V$  is given by

$$A_V^* h(y) = \frac{q^d}{|V|^2} \sum_{x \in V} V(x - y) h(x)$$

where  $h : (V, \sigma) \rightarrow \mathbb{C}$  and  $y \in (\mathbb{F}_q^d, dx)$ . Observe that if  $V = -V := \{x \in \mathbb{F}_q^d : -x \in V\}$ , then

$$A_V^* h = \frac{q^{2d}}{|V|^2} (hV) * V.$$

By duality, if  $1 < p, r < \infty$ , then the estimate (1.1) implies that

$$(1.2) \quad \|A_V^* h\|_{L^{p'}(\mathbb{F}_q^d, dx)} \leq C \|h\|_{L^r(V, \sigma)} \quad \text{for all } h : V \rightarrow \mathbb{C},$$

where  $p' = p/(p-1)$  and  $r' = r/(r-1)$ . We define  $A_V^*(r' \rightarrow p')$  as the best constant  $C > 0$  such that the estimate (1.2) holds. It follows that  $A_V(p \rightarrow r) \lesssim 1 \iff A_V^*(r' \rightarrow p') \lesssim 1$ .

**1.3. Statement of main results.** Results on the restricted averaging problem are based on geometric properties of the underlying variety  $V \subset \mathbb{F}_q^d$ . The structure of the variety  $V$  can be explained in terms of the inverse Fourier transform of the normalized surface measure  $\sigma$  on  $V$ . Recall that the inverse Fourier transform  $\sigma^\vee$  of the surface measure  $\sigma$  is defined by

$$\sigma^\vee(m) = \int_V \chi(m \cdot x) d\sigma(x) = \frac{1}{|V|} \sum_{x \in V} \chi(m \cdot x) \quad \text{for } m \in (\mathbb{F}_q^d, dm).$$

We shall derive certain results on varieties which possess general properties of hypersurfaces such as spheres, paraboloids, or cones in finite fields. To precisely state our main results, we need to classify varieties  $V$  according to their Fourier decay.

**Definition 1.3.** An algebraic variety  $V \subset (\mathbb{F}_q^d, dx)$  will be called a regular variety if  $|V| \sim q^{d-1}$  and  $|\sigma^\vee(m)| \lesssim q^{-(d-1)/2}$  for all  $m \in \mathbb{F}_q^d \setminus (0, \dots, 0)$ , where  $\sigma$  denotes the normalized surface measure on the variety  $V$ .

As the first main result, we obtain the sharp mapping properties of the restricted averaging operator to a regular variety (see Figure 1).

**Theorem 1.4.** Let  $\sigma$  be the normalized surface measure on a regular variety  $V \subset (\mathbb{F}_q^d, dx)$ . Then we have  $A_V(p \rightarrow r) \lesssim 1$  if and only if  $(1/p, 1/r)$  lies on the convex hull of points  $(0, 0)$ ,  $(0, 1)$ ,  $((d-1)/d, 1)$ , and  $((d-1)/d, 1/d)$ .

It is well known ([10, 16]) that typical examples of regular varieties in  $\mathbb{F}_q^d$  are the sphere  $S_j := \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \dots + x_d^2 = j \neq 0\}$  and the paraboloid  $P := \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \dots + x_{d-1}^2 = x_d\}$ . Thus, Theorem 1.4 provides us with the best result on the restricted averaging problem for the sphere  $S_j$  and the paraboloid  $P$ . However, if a variety  $V \subset \mathbb{F}_q^d$  is not a regular variety, then it may not be simple

to prove the sharp  $L^p \rightarrow L^r$  estimates of the restricted averaging operator, because the variety  $V$  may contain a large dimensional affine subspace which no longer has any curvature. Recall that a cone  $C$  in  $\mathbb{F}_q^d$  is defined as

$$(1.3) \quad C = \{x \in \mathbb{F}_q^d : x_1^2 + x_2^2 + \cdots + x_{d-2}^2 = x_{d-1}x_d\}.$$

The regular property of the cone  $C \subset \mathbb{F}_q^d$  depends on the dimension  $d \geq 3$ . In fact, we shall see from Corollary 4.4 in Section 2 that the cone  $C \subset \mathbb{F}_q^d$  is not a regular variety in even dimensions  $d \geq 4$  but the cone  $C$  is a regular variety in odd dimensions  $d \geq 3$ . Hence, if the dimension,  $d \geq 3$ , is odd, then the sharp  $L^p \rightarrow L^r$  estimates for the operator  $A_C$  follows immediately from Theorem 1.4. For this reason, we shall focus on studying this problem for the cone in even dimensions  $d \geq 4$ . In this paper, except for endpoints, we shall establish the sharp mapping properties of the restricted averaging operator to the cone  $C$  in even dimensions. More precisely we have the following result (see Figure 1).

**Theorem 1.5.** *Let  $\sigma_c$  be the normalized surface measure on the cone  $C \subset \mathbb{F}_q^d$  defined as in (1.3). Denote by  $\Omega$  the convex hull of points  $(0, 0), (0, 1), ((d-1)/d, 1)$ ,*

$$P_1 := \left( \frac{d-1}{d}, \frac{1}{d-2} \right) \text{ and } P_2 := \left( \frac{d^2-3d+2}{d^2-2d+2}, \frac{d-2}{d^2-2d+2} \right).$$

*If the dimension,  $d \geq 4$ , is even, then we have the following results:*

- (1) *If  $(1/p, 1/r) \in \Omega \setminus \{P_1, P_2\}$ , then  $A_C(p \rightarrow r) \lesssim 1$*
- (2) *If  $-1 \in \mathbb{F}_q$  is a square number and  $A_C(p \rightarrow r) \lesssim 1$ , then  $(1/p, 1/r) \in \Omega$ .*
- (3) *If we put  $P_1 = (1/p, 1/r)$ , then the following restricted type inequality holds:*

$$\|f * \sigma_c\|_{L^r(C, \sigma_c)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)} \quad \text{for all characteristic functions } f : \mathbb{F}_q^d \rightarrow \mathbb{C}.$$

- (4) *If we put  $P_2 = (1/p, 1/r)$ , then the weak-type estimate*

$$\|f * \sigma_c\|_{L^{r, \infty}(C, \sigma_c)} \lesssim \|f\|_{L^p(\mathbb{F}_q^d, dx)}$$

*holds.*

## 2. NECESSARY CONDITIONS

Let  $V \subset \mathbb{F}_q^d$  be an algebraic variety with  $|V| \sim q^{d-1}$ . We denote by  $\sigma$  the normalized surface measure on  $V$ . Then we have the following necessary conditions for the boundedness of  $A_V(p \rightarrow r)$ .

**Lemma 2.1.** *Let  $1 \leq p, r \leq \infty$ . Assume that  $A_V(p \rightarrow r) \lesssim 1$ . Then we must have*

$$(2.1) \quad \frac{1}{p} \leq \frac{d-1}{d} \quad \text{and} \quad \frac{1}{p(d-1)} \leq \frac{1}{r}.$$

*In addition, if we assume that  $V$  contains an affine subspace  $\Pi$  with  $|\Pi| = q^\alpha$ , then we must have*

$$(2.2) \quad \frac{d-\alpha}{p(d-1-\alpha)} \leq \frac{1}{r} + 1.$$

*Proof.* For each  $\mathbf{a} \in \mathbb{F}_q^d$ , define

$$\delta_{\mathbf{a}}(x) = \begin{cases} 1 & \text{if } x = \mathbf{a} \\ 0 & \text{if } x \neq \mathbf{a}. \end{cases}$$

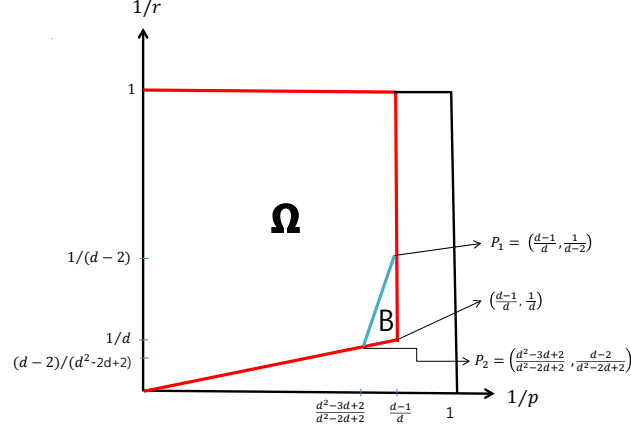


FIGURE 1. The region  $\Omega \cup B$  is related to Theorem 1.4 which gives optimal  $L^p \rightarrow L^r$  result for regular varieties. On the other hand, the region  $\Omega \setminus \{P_1, P_2\}$  indicates the conclusion of Theorem 1.5 which provides us of the sharp results except for two endpoints  $P_1, P_2$  for the cone in even dimensions  $d \geq 4$  provided that  $-1 \in \mathbb{F}_q$  is a square number.

If we test (1.1) with  $f$  equal to  $\delta_0$ , we obtain

$$\|\delta_0 * \sigma\|_{L^r(V, \sigma)} \lesssim \|\delta_0\|_{L^p(\mathbb{F}_q^d, dx)}.$$

Since  $\|\delta_0\|_{L^p(\mathbb{F}_q^d, dx)} = q^{-\frac{d}{p}}$  and  $\|\delta_0 * \sigma\|_{L^r(V, \sigma)} = 1/|V| \sim q^{-(d-1)}$ , we obtain a necessary condition  $1/p \leq (d-1)/d$  for the boundedness of  $A_V(p \rightarrow r)$ . In order to obtain another necessary condition  $1/(p(d-1)) \leq 1/r$ , we shall use the estimate (1.2). If we test (1.2) with  $h$  equal to  $\delta_a$  for some  $a \in V$ , we must have

$$\|A_V^* \delta_a\|_{L^{p'}(\mathbb{F}_q^d, dx)} \lesssim \|\delta_a\|_{L^{r'}(V, \sigma)}.$$

Notice that for  $x \in (\mathbb{F}_q^d, dx)$ , we have

$$A_V^* \delta_a(x) = \frac{q^d}{|V|^2} \sum_{y \in V} V(y-x) \delta_a(y) = \frac{q^d}{|V|^2} V(a-x).$$

It is not hard to see that

$$\|A_V^* \delta_a\|_{L^{p'}(\mathbb{F}_q^d, dx)} = \left(\frac{|V|}{q^d}\right)^{1/p'} \frac{q^d}{|V|^2} \sim q^{-d+2-1/p'}$$

and

$$\|\delta_a\|_{L^{r'}(V, \sigma)} = |V|^{-1/r'} \sim q^{-(d-1)/r'}.$$

Thus, we must have  $-d+2-1/p' \leq -(d-1)/r'$  which yields another necessary condition  $1/(p(d-1)) \leq 1/r$  for the boundedness of  $A_V(p \rightarrow r)$ . Now we prove the necessary condition (2.2). Assume that the variety  $V$  contains an affine subspace

$\Pi$  with  $|\Pi| = q^\alpha$ . If we test (1.2) with  $h$  equal to the characteristic function on  $\Pi$ , we obtain

$$(2.3) \quad \|A_V^* \Pi\|_{L^{p'}(\mathbb{F}_q^d, dx)} \lesssim \|\Pi\|_{L^{r'}(V, \sigma)}.$$

It is clear that

$$(2.4) \quad \|\Pi\|_{L^{r'}(V, \sigma)} = \left( \frac{|\Pi|}{|V|} \right)^{1/r'} \sim q^{(\alpha-d+1)/r'}.$$

Since  $\Pi \subset V$ , we see that  $A_V^* \Pi(x) = \frac{q^d}{|V|^2} \sum_{y \in \Pi} V(y-x)$  for  $x \in (\mathbb{F}_q^d, dx)$ . It follows

$$\|A_V^* \Pi\|_{L^{p'}(\mathbb{F}_q^d, dx)}^{p'} = \frac{q^{dp'-d}}{|V|^{2p'}} \sum_{x \in \mathbb{F}_q^d} \left( \sum_{y \in \Pi} V(y-x) \right)^{p'}.$$

Since  $\Pi \subset V$  is an affine subspace, we can choose a set  $M \subset \mathbb{F}_q^d$  such that  $|M| = |\Pi|$  and  $\sum_{y \in \Pi} V(y-x) = |\Pi|$  for all  $x \in M$ . Thus, we see that

$$\|A_V^* \Pi\|_{L^{p'}(\mathbb{F}_q^d, dx)}^{p'} \geq \frac{q^{dp'-d} |\Pi|^{1+p'}}{|V|^{2p'}} \sim \frac{q^{dp'-d} q^{\alpha(1+p')}}{q^{2p'(d-1)}} = q^{(\alpha-d+2)p'+\alpha-d},$$

where we used the conditions that  $|\Pi| = q^\alpha$  and  $|V| \sim q^{d-1}$ . It follows that

$$\|A_V^* \Pi\|_{L^{p'}(\mathbb{F}_q^d, dx)} \gtrsim q^{\alpha-d+2+(\alpha-d)/p'}.$$

From this inequality, (2.3), and (2.4), we must have

$$\alpha - d + 2 + (\alpha - d)/p' \leq (\alpha - d + 1)/r'$$

which yields the necessary condition (2.2) for the boundedness of  $A_V(p \rightarrow r)$ .  $\square$

### 3. PROOF OF THEOREM 1.4

In this section, we shall prove Theorem 1.4. The necessary part for the boundedness of  $A_V(p \rightarrow r)$  follows immediately from a direct consequence of (2.1) in Lemma 2.1. It remains to prove the sufficient condition for the boundedness of  $A_V(p \rightarrow r)$ . To prove this, observe that  $A_V(\infty \rightarrow \infty) \lesssim 1$ . Now, By the Riesz-Thorin interpolation theorem (see Theorem 1.7 in [1]) and Remark 1.2, it will be enough to show

$$A_V \left( \frac{d}{d-1} \rightarrow d \right) \lesssim 1.$$

Thus, our task is to establish the following estimate

$$(3.1) \quad \|f * \sigma\|_{L^d(V, \sigma)} \lesssim \|f\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \text{ for all } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}.$$

For each  $m \in (\mathbb{F}_q^d, dm)$ , define  $K(m) = \sigma^\vee(m) - \delta_0(m)$ . Then the measure  $\sigma$  can be identified with the function  $\sigma(x) = \widehat{K}(x) + \widehat{\delta}_0(x) = \widehat{K}(x) + 1$  for  $x \in (\mathbb{F}_q^d, dx)$ . To obtain the estimate (3.1), it suffices to prove the following estimates:

$$(3.2) \quad \|f * 1\|_{L^d(V, \sigma)} \lesssim \|f\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \text{ for all } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C},$$

$$(3.3) \quad \|f * \widehat{K}\|_{L^d(V, \sigma)} \lesssim \|f\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \text{ for all } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}.$$

Since  $\max_{x \in V} |f * 1(x)| \leq \|f\|_{L^1(\mathbb{F}_q^d, dx)}$ , the estimate (3.2) follows by observing

$$\|f * 1\|_{L^d(V, \sigma)} \leq \|f\|_{L^1(\mathbb{F}_q^d, dx)} \|1\|_{L^d(V, \sigma)} \leq \|f\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)}.$$

Now notice that (3.3) can be obtained by interpolating the following estimates:

$$(3.4) \quad \|f * \widehat{K}\|_{L^\infty(V, \sigma)} \lesssim q \|f\|_{L^1(\mathbb{F}_q^d, dx)} \text{ for all } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}.$$

and

$$(3.5) \quad \|f * \widehat{K}\|_{L^2(V, \sigma)} \lesssim q^{\frac{-d+2}{2}} \|f\|_{L^2(\mathbb{F}_q^d, dx)} \text{ for all } f : (\mathbb{F}_q^d, dx) \rightarrow \mathbb{C}.$$

Thus, our task is to show that (3.4) and (3.5) hold. Let us prove (3.4). Observe

$$\max_{y \in \mathbb{F}_q^d} |\widehat{K}(y)| = \max_{y \in \mathbb{F}_q^d} |\sigma(y) - 1| = \max_{y \in \mathbb{F}_q^d} \left| \frac{q^d V(y)}{|V|} - 1 \right| \leq \frac{q^d}{|V|} \sim q.$$

Then the estimate (3.4) follows by observing that for any  $x \in V$ ,

$$|f * \widehat{K}(x)| \leq \left( \max_{y \in \mathbb{F}_q^d} |\widehat{K}(y)| \right) \frac{1}{q^d} \sum_{y \in \mathbb{F}_q^d} |f(x - y)| \lesssim q \|f\|_{L^1(\mathbb{F}_q^d, dx)}.$$

Finally, we shall prove the estimate (3.5). From the definition of the function  $K$  and the assumption on a regular variety  $V$ , we see that

$$(3.6) \quad \max_{m \in \mathbb{F}_q^d} |K(m)| \lesssim q^{-\frac{(d-1)}{2}},$$

which shall be used to prove (3.5). In addition, we shall use the following restriction estimate.

**Lemma 3.1.** *Let  $\sigma$  be the normalized surface measure on a variety  $V \subset (\mathbb{F}_q^d, dx)$  with  $|V| \sim q^{d-1}$ . Then we have*

$$\|\widehat{g}\|_{L^2(V, \sigma)} \lesssim q^{\frac{1}{2}} \|g\|_{L^2(\mathbb{F}_q^d, dm)} \text{ for all } g : (\mathbb{F}_q^d, dm) \rightarrow \mathbb{C}.$$

*Proof.* By duality, it suffices to prove the following extension estimate:

$$\|(f\sigma)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} \sim q^{\frac{1}{2}} \|f\|_{L^2(V, \sigma)} \text{ for all } f : V \rightarrow \mathbb{C}.$$

Since  $\sigma(x) = \frac{q^d}{|V|} V(x)$ , it follows from Plancherel's theorem that

$$\begin{aligned} \|(f\sigma)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} &= \frac{q^d}{|V|} \|(fV)^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = \frac{q^d}{|V|} \|fV\|_{L^2(\mathbb{F}_q^d, dx)} \\ &= \frac{q^{d/2}}{|V|^{1/2}} \|f\|_{L^2(V, \sigma)} \sim q^{\frac{1}{2}} \|f\|_{L^2(V, \sigma)}. \end{aligned}$$

□

To complete the proof of the estimate (3.5), we write  $\|f * \widehat{K}\|_{L^2(V, \sigma)} = \|\widehat{f^\vee K}\|_{L^2(V, \sigma)}$ , and apply Lemma 3.1 and (3.6). Then we see that

$$\begin{aligned} \|f * \widehat{K}\|_{L^2(V, \sigma)} &= \|\widehat{f^\vee K}\|_{L^2(V, \sigma)} \lesssim q^{1/2} \|f^\vee K\|_{L^2(\mathbb{F}_q^d, dm)} \\ &\lesssim q^{1/2} q^{-(d-1)/2} \|f^\vee\|_{L^2(\mathbb{F}_q^d, dm)} = q^{(-d+2)/2} \|f\|_{L^2(\mathbb{F}_q^d, dx)}, \end{aligned}$$

where Plancherel's theorem was used to obtain the last equality. Thus, our proof is complete.



## 4. PROPERTIES OF THE CONE

Recall that the cone  $C \subset \mathbb{F}_q^d$  is defined as the set

$$\{x \in \mathbb{F}_q^d : x_{d-1}x_d = x_1^2 + x_2^2 + \cdots + x_{d-2}^2\}$$

and  $\sigma_c$  denotes the normalized surface measure on the cone  $C$ . In this section, we collect preliminary lemmas which play an important role in proving Theorem 1.5. Observing the decay of the inverse Fourier transform on the cone, one may analyze the structural features of the cone. As we shall see below, the inverse Fourier transform on the cone is closely related to the classical Gauss sum. Recall that the Gauss sum  $G$  is defined by

$$G = \sum_{t \in \mathbb{F}_q} \eta(t) \chi(t) \text{ and } |G| = q^{\frac{1}{2}}$$

where  $\eta$  denotes the quadratic character of  $\mathbb{F}_q^*$ . Since  $\sum_{t \in \mathbb{F}_q} \chi(at^2) = G\eta(a)$  for  $a \in \mathbb{F}_q^*$ , Completing the square and using the change of variables, we see

$$(4.1) \quad \sum_{t \in \mathbb{F}_q} \chi(at^2 + bt) = G\eta(t) \chi\left(\frac{b^2}{-4a}\right) \quad \text{for } a \in \mathbb{F}_q^*, b \in \mathbb{F}_q.$$

The inverse Fourier transform on the cone can be explicitly expressed.

**Lemma 4.1.** *Let  $C \subset \mathbb{F}_q^d$  be the cone. For each  $\ell \in \mathbb{F}_q$  and  $\xi = (\xi_1, \xi_2, \dots, \xi_d) \in \mathbb{F}_q^d$ , define  $\Gamma_\ell(\xi) = \xi_1^2 + \xi_2^2 + \cdots + \xi_{d-2}^2 - \ell\xi_{d-1}\xi_d$ . Then we have the following results:*

(1) *If the dimension,  $d \geq 4$ , is even, then*

$$C^\vee(m) = \begin{cases} \frac{\delta_0(m)}{q} + \frac{(q-1)G^{d-2}}{q^d} & \text{for } \Gamma_4(m) = 0 \\ \frac{-G^{d-2}}{q^d} & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

(2) *If the dimension,  $d \geq 3$ , is odd, then*

$$C^\vee(m) = \begin{cases} \frac{\delta_0(m)}{q} & \text{for } \Gamma_4(m) = 0 \\ \frac{G^{d-1}}{q^d} \eta(-\Gamma_4(m)) & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

*Proof.* Notice that  $C = \{x \in \mathbb{F}_q^d : \Gamma_1(x) = 0\}$ . By the definition of the inverse Fourier transform and the orthogonality relation of  $\chi$ , we see that

$$\begin{aligned} C^\vee(m) &= q^{-d} \sum_{x \in C} \chi(m \cdot x) = q^{-d} \sum_{x \in \mathbb{F}_q^d : \Gamma_1(x)=0} \chi(m \cdot x) \\ &= q^{-d} \sum_{x \in \mathbb{F}_q^d} \left( q^{-1} \sum_{s \in \mathbb{F}_q} \chi(s\Gamma_1(x)) \right) \chi(m \cdot x) \\ &= \frac{\delta_0(m)}{q} + q^{-d-1} \sum_{s \neq 0} \sum_{x \in \mathbb{F}_q^d} \chi(s\Gamma_1(x) + \chi(m \cdot x)) \\ &= \frac{\delta_0(m)}{q} + \frac{1}{q^{d+1}} \sum_{s \neq 0} \sum_{x \in \mathbb{F}_q^d} \chi(s(x_1^2 + \cdots + x_{d-2}^2 - x_{d-1}x_d)) \chi(m \cdot x) \end{aligned}$$

By the formula (4.1) we see that

$$C^\vee(m) = \frac{\delta_0(m)}{q} + \frac{G^{d-2}}{q^{d+1}} \sum_{s \neq 0} \eta^{d-2}(s) \chi\left(\frac{m_1^2 + \cdots + m_{d-2}^2}{-4s}\right) I(m_{d-1}, m_d),$$

where we define

$$I(m_{d-1}, m_d) = \sum_{x_{d-1} \in \mathbb{F}_q} \chi(m_{d-1}x_{d-1}) \sum_{x_d \in \mathbb{F}_q} \chi((-sx_{d-1} + m_d)x_d).$$

Compute the sum over  $x_d \in \mathbb{F}_q$  by the orthogonality relation of  $\chi$  and obtain that

$$C^\vee(m) = \frac{\delta_0(m)}{q} + \frac{G^{d-2}}{q^d} \sum_{s \neq 0} \eta^{d-2}(s) \chi\left(\frac{m_1^2 + \cdots + m_{d-2}^2 - 4m_{d-1}m_d}{-4s}\right).$$

Since  $\eta^{d-2} = 1$  for even  $d \geq 4$ , the first statement of Lemma 4.1 follows. To prove the second part of Lemma 4.1, we first note that if the dimension  $d \geq 3$ , is odd, then  $\eta^{d-2}(s) = \eta(s) = \eta(s^{-1})$  for  $s \neq 0$ . Therefore, if  $m_1^2 + \cdots + m_{d-2}^2 - 4m_{d-1}m_d = 0$ , the statement follows immediately from the orthogonality relation of  $\eta$ . On the other hand, if  $m_1^2 + \cdots + m_{d-2}^2 - 4m_{d-1}m_d \neq 0$ , then the statement follows from a change of variables, the definition of the Gauss sum, and properties of the quadratic character  $\eta$ .  $\square$

We need the following lemma.

**Lemma 4.2.** *Let  $C \subset \mathbb{F}_q^d$  be the cone. If  $d \geq 3$ , then we have  $|C| \sim q^{d-1}$ .*

*Proof.* From the definition of the inverse Fourier transform on the cone  $C \subset \mathbb{F}_q^d$  and the conclusion of Lemma 4.1, we see that

$$C^\vee(0, \dots, 0) = \frac{|C|}{q^d} = \begin{cases} q^{-1} + \frac{(q-1)G^{d-2}}{q^d} & \text{for even } d \geq 4 \\ q^{-1} & \text{for odd } d \geq 3. \end{cases}$$

Since the absolute value of the Gauss sum is  $\sqrt{q}$  (namely,  $|G| = \sqrt{q}$ ), we conclude

$$|C| \sim q^{d-1} \quad \text{for } d \geq 3.$$

$\square$

Since  $|G| = \sqrt{q}$ , the following result is immediate from Lemma 4.1.

**Corollary 4.3.** *Let  $C \subset \mathbb{F}_q^d$  be the cone. Assume that  $m \in \mathbb{F}_q^d \setminus \{(0, \dots, 0)\}$ . Then the following two statements hold:*

(1) *If the dimension,  $d \geq 4$ , is even, then*

$$|C^\vee(m)| \sim \begin{cases} q^{-\frac{d}{2}} & \text{for } \Gamma_4(m) = 0 \\ q^{-\frac{(d+2)}{2}} & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

(2) *If the dimension,  $d \geq 3$ , is odd, then*

$$|C^\vee(m)| = \begin{cases} 0 & \text{for } \Gamma_4(m) = 0 \\ q^{-\frac{(d+1)}{2}} & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

The following result can be obtained by using Corollary 4.3.

**Corollary 4.4.** *Let  $\sigma_c$  be the normalized surface measure on the cone  $C \subset \mathbb{F}_q^d$ . Suppose that  $(0, \dots, 0) \neq m \in \mathbb{F}_q^d$ . Then we have the following facts:*

(1) *If the dimension,  $d \geq 4$ , is even, then*

$$|\sigma_c^\vee(m)| \sim \begin{cases} q^{-\frac{(d-2)}{2}} & \text{for } \Gamma_4(m) = 0 \\ q^{-\frac{d}{2}} & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

(2) If the dimension,  $d \geq 3$ , is odd, then

$$|\sigma_c^\vee(m)| = \begin{cases} 0 & \text{for } \Gamma_4(m) = 0 \\ q^{-\frac{(d-1)}{2}} & \text{for } \Gamma_4(m) \neq 0. \end{cases}$$

*Proof.* Since  $\sigma_c^\vee(m) = \frac{q^d}{|C|} C^\vee(m)$  and  $|C| \sim q^{d-1}$  for  $d \geq 3$ , the statement follows immediately from Corollary 4.3.  $\square$

We also need the following result.

**Lemma 4.5.** *Let  $C^* = \{m \in \mathbb{F}_q^d : \Gamma_4(m) = 0\}$ . If the dimension,  $d \geq 4$ , is even, then we have*

$$\sum_{m \in C^*} |E^\vee(m)|^2 \lesssim q^{-d-1} |E| + q^{-\frac{3d}{2}} |E|^2 \quad \text{for all } E \subset (\mathbb{F}_q^d, dx).$$

*Proof.* It is clear that  $|C^*| \sim q^{d-1}$  for even  $d \geq 4$ , because  $|C^*| = |C| \sim q^{d-1}$ . It follows that

$$\begin{aligned} \sum_{m \in C^*} |E^\vee(m)|^2 &= q^{-2d} \sum_{x, y \in E} \sum_{m \in C^*} \chi(m \cdot (x - y)) \\ &= q^{-2d} \sum_{x, y \in E: x=y} |C^*| + q^{-2d} \sum_{x, y \in E: x \neq y} \sum_{m \in C^*} \chi(m \cdot (x - y)) \\ &\lesssim q^{-2d} q^{d-1} |E| + q^{-2d} |E|^2 \max_{\beta \neq (0, \dots, 0)} \left| \sum_{m \in C^*} \chi(m \cdot \beta) \right|. \end{aligned}$$

Adapting the arguments used to prove the first part of Lemma 4.1, we see that

$$\max_{\beta \neq (0, \dots, 0)} \left| \sum_{m \in C^*} \chi(m \cdot \beta) \right| \lesssim q^{\frac{d}{2}}.$$

Combining this with the above estimate, we complete the proof.  $\square$

## 5. PROOF OF THEOREM 1.5

To prove the statement (2) of Theorem 1.5, it suffices by Lemma 2.1 to show that if  $-1$  is a square number and the dimension,  $d \geq 4$ , is even, then the cone  $C \subset \mathbb{F}_q^d$  contains a subspace  $\Pi$  such that  $|\Pi| = q^{d/2}$ . Now, define

$$\Pi = \{(t_1, it_1, \dots, t_{(d-2)/2}, it_{(d-2)/2}, s, 0) \in \mathbb{F}_q^d : s, t_j \in \mathbb{F}_q, j = 1, 2, \dots, (d-2)/2\},$$

where  $i$  denotes an element of  $\mathbb{F}_q$  such that  $i^2 = -1$ . It is clear that  $\Pi$  is a  $d/2$ -dimensional subspace contained in the cone  $C$ . Thus, we complete the proof of the statement (2) of Theorem 1.5.

Next, notice that if the statements (3), (4) of Theorem 1.5 are true, then the statement (1) of Theorem 1.5 follows from Remark 1.2 and the Marcinkiewicz interpolation theorem (see Theorem 4.13 in [1]). In conclusion, to complete the proof of Theorem 1.5 it remains to show that the statements (3),(4) of Theorem 1.5 hold, which shall be proved in the following subsections.

**5.1. Proof of the statement (3) of Theorem 1.5.** We aim to prove that if the dimension,  $d \geq 4$ , is even, then

$$\|E * \sigma_c\|_{L^{d-2}(C, \sigma_c)} \lesssim \|E\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx).$$

As before, we can write  $\sigma_c = \widehat{K} + 1$ , where we define  $K(m) = \sigma_c^\vee(m) - \delta_0(m)$  for  $m \in (\mathbb{F}_q^d, dm)$ . It suffices to prove that

$$(5.1) \quad \|E * 1\|_{L^{d-2}(C, \sigma_c)} \lesssim \|E\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx)$$

and

$$(5.2) \quad \|E * \widehat{K}\|_{L^{d-2}(C, \sigma_c)} \lesssim \|E\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx).$$

Notice that  $\max_{x \in C} |E * 1(x)| \leq \|E\|_{L^1(\mathbb{F}_q^d, dx)}$ . Then the inequality (5.1) follows because

$$\|E * 1\|_{L^{d-2}(C, \sigma_c)} \leq \|E\|_{L^1(\mathbb{F}_q^d, dx)} \|1\|_{L^{d-2}(C, \sigma_c)} = \|E\|_{L^1(\mathbb{F}_q^d, dx)} \leq \|E\|_{L^{d/(d-1)}(\mathbb{F}_q^d, dx)}.$$

Notice that the estimate (5.2) can be obtained by interpolating the following two inequalities:

$$(5.3) \quad \|E * \widehat{K}\|_{L^\infty(C, \sigma_c)} \lesssim q \|E\|_{L^1(\mathbb{F}_q^d, dx)} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx)$$

and

$$(5.4) \quad \|E * \widehat{K}\|_{L^2(C, \sigma_c)} \lesssim q^{-\frac{(d-4)}{2}} \|E\|_{L^{2d/(d+2)}(\mathbb{F}_q^d, dx)} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx)$$

The inequality (5.3) follows from the arguments used to prove the estimate (3.4). Now we prove the inequality (5.4). By the property of convolution functions and Lemma 3.1, we can write

$$\|E * \widehat{K}\|_{L^2(C, \sigma_c)} = \|\widehat{E^\vee K}\|_{L^2(C, \sigma_c)} \lesssim q^{1/2} \|E^\vee K\|_{L^2(\mathbb{F}_q^d, dm)}.$$

Thus, to prove (5.4) it will be enough to show that

$$(5.5) \quad \begin{aligned} \|E^\vee K\|_{L^2(\mathbb{F}_q^d, dm)}^2 &\lesssim q^{-d+3} \|E\|_{L^{2d/(d+2)}(\mathbb{F}_q^d, dx)}^2 \\ &= \frac{|E|^{(d+2)/d}}{q^{2d-1}} \quad \text{for all } E \subset (\mathbb{F}_q^d, dx). \end{aligned}$$

Observe by the definition of  $K$  that  $K(0, \dots, 0) = 0$  and  $K(m) = \sigma_c^\vee(m)$  for  $m \neq (0, \dots, 0)$ . Let us prove the estimate (5.5).

**(Case I)** Assume that  $|E| \geq q^{d/2}$  for even  $d \geq 4$ . Applying (1) of Corollary 4.4 and Plancherel's theorem, we obtain

$$\|E^\vee K\|_{L^2(\mathbb{F}_q^d, dm)}^2 = \sum_{m \in \mathbb{F}_q^d} |E^\vee(m)|^2 |K(m)|^2 \lesssim q^{-d+2} q^{-d} |E| = q^{-2d+2} |E|.$$

Since  $q^{-2d+2} |E| \leq q^{-2d+1} |E|^{(d+2)/d}$  for  $|E| \geq q^{d/2}$ , the estimate (5.5) holds.

(**Case II**) Assume that  $|E| \leq q^{d/2}$  for even  $d \geq 4$ . Using (1) of Corollary 4.4 and Lemma 4.5, we see that

$$\begin{aligned}
\|E^\vee K\|_{L^2(\mathbb{F}_q^d, dm)}^2 &= \sum_{m \in \mathbb{F}_q^d} |E^\vee(m)|^2 |K(m)|^2 \\
&= \sum_{\Gamma_4(m)=0} |E^\vee(m)|^2 |K(m)|^2 + \sum_{\Gamma_4(m) \neq 0} |E^\vee(m)|^2 |K(m)|^2 \\
&\lesssim q^{-d+2} \left( q^{-d-1} |E| + q^{-\frac{3d}{2}} |E|^2 \right) + q^{-d} \sum_{m \in \mathbb{F}_q^d} |E^\vee(m)|^2 \\
&= q^{-2d+1} |E| + q^{(-5d+4)/2} |E|^2 + q^{-2d} |E| \\
&\sim q^{-2d+1} |E| + q^{(-5d+4)/2} |E|^2.
\end{aligned}$$

Since the quantity in the last line is  $\lesssim q^{-2d+1} |E|^{(d+2)/d}$  if  $|E| \leq q^{d/2}$ , the estimate (5.5) holds. We have completed the proof of the statement (3) of Theorem 1.5.

**5.2. Proof of the statement (4) of Theorem 1.5.** By duality, it suffices to prove the following restricted type estimate:

$$\|A_C^* F\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} \lesssim \|F\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c),$$

where we recall that

$$A_C^* F(x) = \frac{q^d}{|C|^2} \sum_{y \in C} C(y-x) F(y).$$

Since  $F \subset C$  and  $C = -C$ , we see that  $A_C^* F = \frac{q^{2d}}{|C|^2} F * C$ . Hence, our task is to prove that

$$\left\| \frac{q^{2d}}{|C|^2} F * C \right\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} \lesssim \|F\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c).$$

For each  $m \in (\mathbb{F}_q^d, dm)$ , define  $H(m) = C^\vee(m) - \frac{|C|}{q^d} \delta_{\mathbf{0}}(m)$ . Then we can write  $C(x) = \widehat{H}(x) + \frac{|C|}{q^d}$  for  $x \in (\mathbb{F}_q^d, dx)$ . To complete the proof, it is enough to show

$$(5.6) \quad \left\| \frac{q^d}{|C|} F * 1 \right\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} \lesssim \|F\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c)$$

and

$$(5.7) \quad \left\| \frac{q^{2d}}{|C|^2} F * \widehat{H} \right\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} \lesssim \|F\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c).$$

Since  $F * 1(x) = \frac{|F|}{q^d}$  for all  $x \in (\mathbb{F}_q^d, dx)$ , the inequality (5.6) follows by observing

$$\left\| \frac{q^d}{|C|} F * 1 \right\|_{L^{\frac{d^2-2d+2}{d}}(\mathbb{F}_q^d, dx)} = \frac{|F|}{|C|} \leq \left( \frac{|F|}{|C|} \right)^{\frac{d^2-3d+4}{d^2-2d+2}} = \|F\|_{L^{\frac{d^2-2d+2}{d^2-3d+4}}(C, \sigma_c)}.$$

It remains to prove the inequality (5.7). To do this, we claim that the following two estimates hold:

$$(5.8) \quad \left\| \frac{q^{2d}}{|C|^2} F * \widehat{H} \right\|_{L^\infty(\mathbb{F}_q^d, dx)} \lesssim q \|F\|_{L^1(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c)$$

and

$$(5.9) \quad \left\| \frac{q^{2d}}{|C|^2} F * \widehat{H} \right\|_{L^2(\mathbb{F}_q^d, dx)} \lesssim q^{\frac{-d^2+4d-2}{2d}} \|F\|_{L^{\frac{2d}{d+2}}(C, \sigma_c)} \quad \text{for all } F \subset (C, \sigma_c),$$

which shall be proved below. Notice that the inequality (5.7) is obtained by interpolating (5.8) and (5.9). To obtain the inequality (5.8), we use Young's inequality and observe that  $\|\widehat{H}\|_{L^\infty(\mathbb{F}_q^d, dx)} \lesssim 1$ . Then we see that

$$\begin{aligned} \left\| \frac{q^{2d}}{|C|^2} F * \widehat{H} \right\|_{L^\infty(\mathbb{F}_q^d, dx)} &\leq \frac{q^{2d}}{|C|^2} \|F\|_{L^1(\mathbb{F}_q^d, dx)} \|\widehat{H}\|_{L^\infty(\mathbb{F}_q^d, dx)} \\ &\lesssim \frac{q^d |F|}{|C|^2} = \frac{q^d}{|C|} \|F\|_{L^1(C, \sigma_c)}. \end{aligned}$$

Since  $|C| \sim q^{d-1}$ , the inequality (5.8) follows. Finally, we prove the inequality (5.9). By Plancherel's theorem and the property of convolution functions, we can write

$$\left\| \frac{q^{2d}}{|C|^2} F * \widehat{H} \right\|_{L^2(\mathbb{F}_q^d, dx)} \sim q^2 \|F^\vee H\|_{L^2(\mathbb{F}_q^d, dm)}.$$

Comparing this with the right-hand side of (5.9), we see that it suffices to show

$$(5.10) \quad \begin{aligned} \|F^\vee H\|_{L^2(\mathbb{F}_q^d, dm)}^2 &\lesssim q^{\frac{-d^2-2}{d}} \|F\|_{L^{\frac{2d}{d+2}}(C, \sigma_c)}^2 \\ &\sim \frac{|F|^{(d+2)/d}}{q^{2d+1}} \quad \text{for all } F \subset (C, \sigma_c). \end{aligned}$$

From the definition of  $H$ , we see that  $H(0, \dots, 0) = 0$  and  $H(m) = C^\vee(m)$  for  $m \neq (0, \dots, 0)$ . Let us prove (5.10).

**(Case 1)** Assume that  $|F| \geq q^{d/2}$  for even  $d \geq 4$ . From (1) of Corollary 4.3 and Plancherel's theorem, we see that

$$\|F^\vee H\|_{L^2(\mathbb{F}_q^d, dm)}^2 \lesssim q^{-d} \|F^\vee\|_{L^2(\mathbb{F}_q^d, dm)}^2 = q^{-d} \|F\|_{L^2(\mathbb{F}_q^d, dx)}^2 = q^{-2d} |F|.$$

Since  $q^{-2d} |F| \leq q^{-2d-1} |F|^{(d+2)/d}$  if  $|F| \geq q^{d/2}$ , the inequality (5.10) holds in this case.

**(Case 2)** Assume that  $|F| \leq q^{d/2}$  for even  $d \geq 4$ . From (1) of Corollary 4.3 and Lemma 4.5, it follows that

$$\begin{aligned} \|F^\vee H\|_{L^2(\mathbb{F}_q^d, dm)}^2 &= \sum_{m \in \mathbb{F}_q^d} |F^\vee(m)|^2 |H(m)|^2 \\ &\lesssim q^{-d} \sum_{\Gamma_4(m)=0} |F^\vee(m)|^2 + q^{-d-2} \sum_{\Gamma_4(m) \neq 0} |F^\vee(m)|^2 \\ &\lesssim q^{-d} \left( q^{-d-1} |F| + q^{-\frac{3d}{2}} |F|^2 \right) + q^{-d-2} \sum_{m \in \mathbb{F}_q^d} |F^\vee(m)|^2 \\ &= \left( q^{-2d-1} |F| + q^{-5d/2} |F|^2 \right) + q^{-2d-2} |F| \\ &\sim q^{-2d-1} |F| + q^{-5d/2} |F|^2. \end{aligned}$$

Since the last value is  $\lesssim q^{-2d-1}|F|^{(d+2)/d}$  if  $|F| \leq q^{d/2}$ , the inequality (5.10) also holds. Thus, we complete the proof of the statement (4) of Theorem 1.5.

**Acknowledgement :** The authors would like to thank anonymous referees for their valuable comments which help to improve the manuscript. We also wish to thank David Corvert for fixing grammatical errors and clarifying ambiguous sentences in the previous version of this paper.

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